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Metastability of the *d*-Dimensional Contact Process

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We prove that the d-dimensional supercritical contact process exhibits metastable behavior, in the pathwise sense. This is done by proving the property of thermalization and using Mountford's theorem. We also extend some previous results on the loss of memory of the process.

KEY WORDS: Contact process; metastability; loss of memory; thermalization; infinite-particle systems.

1. INTRODUCTION

This paper is concerned with the contact process for an arbitrary dimension $d \in \mathbb{N} \backslash \{0\}$ restricted to $\Gamma_N = \{-N,..., N\}^d$, $N \in \mathbb{N}$, starting with all sites (in Γ_N) occupied. This is a continuous-time Markov process on $\{0, 1\}^{\Gamma_N}$ with flip rates at a state $\zeta \in \{0, 1\}^{\mathbb{Z}^d}$ given by

$$c(x, \zeta) = \begin{cases} 1 & \text{if } \zeta(x) = 1 \\ \lambda \sum\limits_{\substack{y \in \Gamma_N \\ \|x - y\| = 1}} \zeta(y) & \text{if } \zeta(x) = 0 \end{cases}$$

if $x \in \Gamma_N$ and $c(x, \zeta) = 0$ if $x \in \Gamma_N^c$, where $\lambda \in \mathbb{R}_+$ is a fixed parameter and $\|z\| = \sum_{i=1}^d |z_i|$ for $z \in \mathbb{Z}^d$.

We are interested in the metastable behavior for large N and for $\lambda > \lambda_c(d)$, the critical value for the unrestricted contact process in d

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dimensions, in a sense that we make precise below. The restricted process must eventually die out for all $\lambda \in \mathbb{R}_+$ and we investigate the time it takes for this to happen.

In order to investigate metastability we adopt the method proposed by Cassandro *et al.*,[1] where the pathwise approach was introduced. In that paper they showed the above result for sufficiently large λ in the one-dimensional case.

Informally, the phenomenon of metastability can be described as follows: The system is in a metastable situation if it stays in an apparent state of equilibrium during a long random time and in particular if the statistics of the trajectories stabilizes in this situation, but at the end of a memoryless random time (or asymptotically exponential random time), this statistics has on abrupt break of coherence and stabilize around the true state of equilibrium at $\delta_{\varnothing}$.

The metastable behavior of the supercritical one-dimensional contact process was proven by Schonmann.[2] Moreover, he proved that this does not happen in the subcritical case. Mountford[3] proved that for the d-dimensional supercritical contact process, restricted and initially fully occupied, the suitable normalized time to die out converges to an exponential random variable of mean one as N tends to infinity.

The goal of this paper is to show that the supercritical contact process presents a metastable behavior in any dimension. This is done by proving the property of thermalization for the d-dimensional case, i.e., when the process is out of its equilibrium situation the temporal means stabilize close to the expectation of the statistics of the trajectories in some fixed probability distribution on the configurations of the system. Metastability then follows from Mountford's results[3] on the time for the process to die out.

This extension of the thermalization property to arbitrary dimensions is derived from the fast loss of memory of the process, which we obtain using some results from Bezuidenhout and Grimmett.[4] The concept of loss of memory was implicit in ref. 1 and was later used in Galves *et al.*[5] to study the asymptotic distribution of the time of first occurrence of an anomalous density of particles in a large, fixed region of the space for the supercritical one-dimensional contact process. Here we extend the concept to establish our theorem.

The paper is organized as follows: In Section 2 we introduce the notation to be used in this work and state the metastability theorem. In Section 3 we prove general results about the loss of memory of the contact process, which are used in Section 4 to prove the main theorem. In Section 5 we prove that the phenomenon of metastability does not hold for $\lambda < \lambda_c(d)$.

2. NOTATIONS AND RESULTS

We consider the contact process as being derived from a collection of Poisson processes defined in the sites $x \in \mathbb{Z}^d$, as in the representation introduced by Harris.[6] This representation enables us to couple contact processes with different initial configurations. For a construction and proofs of the existence and uniqueness of the process defined by the parameter λ see, for instance, Durrett[7] and Liggett.[8]

We use the notation $\{\xi^{\eta}_{\Gamma_N}(t)\colon t \geqslant 0\}$ to describe the process restricted to $\Gamma_N \subset \mathbb{Z}^d$ starting from the configuration $\eta \in \{0, 1\}^{\Gamma_N}$, i.e., $\xi^{\eta}_{\Gamma_N}(0) = \eta$. Restricted means that Poisson processes defined in Γ^c_N are not considered.

Given $\eta \in \{0, 1\}^{\Gamma_N}$, we define the hitting time of the trap state, i.e., the empty configuration, starting from η by $T^{\eta}_{\Gamma_N} = \inf\{t \geqslant 0\colon \xi^{\eta}_{\Gamma_N}(t) = \varnothing\}$. For notational convenience we write $\{\xi^{\eta}(t)\colon t \geqslant 0\}$ for the unrestricted process and in this case for the initial condition $\eta \in \{0, 1\}^{\mathbb{Z}^d}$ we define T^{η} analogously.

We call $f\colon \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ a cylindrical function if there is a finite set $\Delta \subset \mathbb{Z}^d$ such that $f(\eta) = f(\eta \cap \Delta)$ for any configuration $\eta \in \{0, 1\}^{\mathbb{Z}^d}$. The support of f is defined as the smallest set of $\mathbb{Z}^d$ which has this property and write $\Delta(f)$ for this set.

Given a real number $b > 0$, $N \in \mathbb{N}$, and a cylindrical function f, we define the temporal mean of f with respect to the process $\{\xi^{\Gamma_N}_{\Gamma_N}(t)\colon t \geqslant 0\}$, as the measure-valued process $\{A^N_b(s, f)\colon s \geqslant 0\}$ given by

$$A^N_b(s, f) = \frac{1}{b} \int_s^{s+b} f(\xi^{\Gamma_N}_{\Gamma_N}(t))\, dt$$

where s is the instant of the beginning of the masurement and b the time interval of observation. We can now state the metastability theorem:

For any $d \geqslant 1$, $\lambda > \lambda_c(d)$, and a cylindrical function f, there is an increasing sequence of positive real numbers $\{b(N, d)\colon N \geqslant 1\}$ such that the process $\{A^N_{b(N,d)}(s\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}], f)\colon s \geqslant 0\}$ converges in distribution, as $N \to \infty$, to a Markovian jump process $\{A(s)\colon s \geqslant 0\}$ defined by

$$A(s) = \begin{cases} \nu & \text{if } \ s < T \\ \delta_{\varnothing} & \text{if } \ s > T \end{cases}$$

where T is an exponential random time of mean one, ν is the extremal non-trivial invariant measure of the unrestricted contact process, and $\delta_{\varnothing}$ is the Dirac δ-measure concentrated in the empty configuration.

3. LOSS OF MEMORY OF THE PROCESS

For any $0 \leqslant q \leqslant 1$ let

$$A(\Gamma_N, q) = \left\{ \eta \in \{0, 1\}^{\mathbb{Z}^d} : \sum_{x \in \Gamma_N} \eta(x) \geqslant q\, |\Gamma_N| \right\}$$

where $|\Gamma_N| = (2N+1)^d$ is the cardinal of the set Γ_N.

In order to obtain our results we will consider contact processes with different initial configurations, using an appropriate coupling established when both processes are constructed in the same probability space following the same collection of Poisson processes considered in the Harris representation.

Lemma 3.1. Let $d \geqslant 1$ and $\lambda > \lambda_c(d)$. For any $0 < q \leqslant 1$ there are $L = L(d, \lambda) \in \mathbb{N}$ and $c = c(d, \lambda) \in (0, 1)$ such that, for every $N \geqslant L$ and every $\eta \in A(\Gamma_N, q)$,

$$\mathbb{P}(T^{\eta \cap \Gamma_N} = \infty) \geqslant 1 - \exp\{-cq(2N+1)^{d-1}\}$$

Proof. The case $d = 1$ follows Theorem (3.29), p. 303 of ref. 8. Suppose $d > 1$ and for $r \in \mathbb{N} \backslash \{0\}$ fixed let $D_r = \{z \in \mathbb{Z}^d : |z_i| \leqslant r,\ i = 1,..., d\}$ be a disc in $\mathbb{Z}^d$.

Bezuidenhout and Grimmett[4] have shown that there is a finite disc $D_r = D$ and a positive integer $L > r$ such that for all $\lambda > \lambda_c(d)$ we have

$$\mathbb{P}\{\xi^D_{[-L, L]^{d-1} \times \mathbb{Z}} \text{ survives in } [0, \infty)\} = \gamma > 0$$

For $N \geqslant L$ and $\eta \in A(\Gamma_N, q)$ fixed, we can find $\varnothing \neq \zeta \subset \eta \cap \Gamma_N$ such that

$$|\zeta| \geqslant \max\left\{ \frac{|\Gamma_N|\, q}{(4L)^{d-1}(2N+1)}, 1 \right\}$$

where $|\zeta|$ is the cardinal of ζ, such that the sets $x + (-L, L)^{d-1} \times \mathbb{Z}$, when $x \in \zeta$, are disjoint. Let x^l for $l = 0, 1,..., |\zeta| - 1$ be an enumeration of the sites of ζ, namely $\zeta = \{x^l \in \eta : l = 0,..., |\zeta| - 1\}$.

Note that there is a strictly lower bound, say $p(d, \lambda, r, h)$, for the probability that at any time $h > 0$, there is a fully occupied disc $D_r + x^l$ in the process $\{\xi^{x^l}_{\Gamma_L + x^l}(t) : 0 \leqslant t \leqslant h\}$, $x^l \in \zeta$, constructed by the coupling described before with the process $\{\xi^\eta(t) : t \geqslant 0\}$. We now show that there is a lower bound for the probability of the event $\{\xi^\zeta(t) \text{ survives in } [0, \infty)\}$.

Defining $B_{d,L}(x^l)=x^l+(-L,L)^{d-1}\times\mathbb{Z}$, both processes

$$\{\xi^{x^l}_{B_{d,L}(x^l)}(t)\colon t\geqslant 0\},\qquad \{\xi^{x^m}_{B_{d,L}(x^m)}(t)\colon t\geqslant 0\},\qquad l\neq m,\quad x^l,x^m\in\zeta$$

which are restricted versions of the process starting from η with the same parameter $\lambda>0$, have independent evolutions.

Denoting by $\{\xi^0(t)\colon t\geqslant 0\}$ the unrestricted process starting at the origin of $\mathbb{Z}^d$, we observe that for each $x^l\in\zeta$ the following inequality holds:

$$\begin{aligned}
&\mathbb{P}\{\xi^{x^l}_{B_{d,L}(x^l)}(t)\text{ survives in }[0,\infty)\}\\
&\quad=\mathbb{P}\{\xi^0_{B_{d,L}(0)}(t)\text{ survives in }[0,\infty)\}\\
&\quad\geqslant\mathbb{P}\{\xi^0_{B_{d,L}(0)}(t)\text{ survives in }[0,\infty),\ \xi^0_{\Gamma_L}(h)=D_r\text{ for some }0\leqslant h\leqslant t\}\\
&\quad=\gamma p(h,r,d,\lambda)>0
\end{aligned}$$

where the first equality follows from the translation invariance property of the contact process and the second from the Markov property.

To complete the proof, note that from the attactiveness of the contact process it follows that

$$\{\xi^{x^l}_{B_{d,L}(x^l)}(t)\text{ survives in }[0,\infty)\}$$

for any $x^l\in\zeta$ implies that

$$\{\xi^{\eta\cap\Gamma_N}(t)\text{ survives in }[0,\infty)\}$$

and by the construction of the process $\{\xi^{x^l}_{B_{d,L}(x^l)}(t)\colon t\geqslant 0\}$ we have that

$$\begin{aligned}
\mathbb{P}\{T^{\eta\cap\Gamma_N}=\infty\}&\geqslant\mathbb{P}\{\xi^{\zeta}(t)\text{ survives in }[0,\infty)\}\\
&\geqslant\mathbb{P}\left\{\bigcup_{l\leqslant|\zeta|-1}\{\xi^{x^l}_{B_{d,L}(x^l)}(t)\text{ survives in }[0,\infty)\}\right\}\\
&=1-\left\{\prod_{l\leqslant|\zeta|-1}\mathbb{P}\{\xi^{x^l}_{B_{d,L}(x^l)}(t)=\varnothing\text{ for some }t>0\}\right\}\\
&\geqslant 1-\prod_{l\leqslant|\zeta|-1}(1-\gamma p(h,r,d,\lambda))\\
&\geqslant 1-\exp\{-\gamma p(h,r,d,\lambda)\,|\zeta|\}\\
&\geqslant 1-\exp\left\{-\gamma p(h,r,d,\lambda)\frac{|\Gamma_N|\,q}{(4L)^{d-1}(2N+1)}\right\}\\
&=1-\exp\{-cq(2N+1)^{d-1}\}
\end{aligned}$$

where $c=\gamma p(h,r,d,\lambda)(4L)^{1-d}\in(0,1)$. ∎

The next lemma estimates the rate, at time t, of the loss of memory of the process as N grows.

Lemma 3.2. Let $d \geqslant 1$ and $\lambda > \lambda_c(d)$. For any $0 < q \leqslant 1$, $N \geqslant 1$, and η and $\zeta \in A(\Gamma_N, q)$, there are $\alpha = \alpha(d, \lambda) \in [1, \infty)$, $c = c(d, \lambda, q) \in (0, \infty)$, and $C = C(d, \lambda) \in [1, \infty)$ such that for every $t \geqslant \alpha N$,

$$\mathbb{P}(\xi^\eta(t) = \xi^\zeta(t) \text{ in } \Gamma_N) \geqslant 1 - C\,|\Gamma_N| \exp\{-ct\}$$

Proof. It is enough to prove the result for $\eta \in A(\Gamma_N, q)$ and $\zeta = \mathbb{Z}^d$.

Given $t \geqslant 1$, it follows from Lemma 3.1 that there are positive constants L and c', independent of N, such that

$$\begin{aligned}
&\mathbb{P}(\exists y \in [-Lt, Lt]^d, \eta(y) = 1, T^y = \infty) \\
&\quad = \mathbb{P}(T^{\eta \cap [-Lt, Lt]^d} = \infty) \\
&\quad \geqslant 1 - \exp\{-c'q(2Lt+1)^{d-1}\}
\end{aligned}$$

Since $(2Lt+1)^{d-1} \geqslant 1 + (d-1)\,Lt$, we obtain that

$$\begin{aligned}
1 - \exp\{-c'q)2Lt+1)^{d-1}\} &\geqslant 1 - \exp\{-c'q(1 + (d-1)\,Lt)\} \\
&= 1 - \exp\{-c'q\} \exp\{-c'q(d-1)\,Lt\} \\
&\geqslant 1 - \exp\{-c'qt\}
\end{aligned}$$

Note that in the case $d = 1$ the inequality above is a consequence of Theorem (3.29), p. 303, of ref. 8.

Define for any $y \in \mathbb{Z}^d$ and $t \geqslant 0$ the coupled region $K_t^y = \xi^y(t) \cup (\xi^{\mathbb{Z}^d}(t))^c$. We have, for any $t \geqslant 1$ fixed, that

$$\begin{aligned}
&\{\exists y \in [-Lt, Lt]^d, \eta(y) = 1, T^y = \infty\} \\
&\quad \subset \bigcup_{y \in \eta} \{\xi^y(t) = \xi^{\mathbb{Z}^d}(t) \text{ in } K_t^y, T^y = \infty\} \\
&\quad \subset \{\xi^\eta(t) = \xi^{\mathbb{Z}^d}(t) \text{ in } \Gamma_N\} \cup \{\Gamma_N \not\subset K_t^y, T^y = \infty\}
\end{aligned}$$

where the last set is for one choice y with the smallest norm.

Therefore

$$\begin{aligned}
&\mathbb{P}(\xi^\eta(t) = \xi^{\mathbb{Z}^d}(t) \text{ in } \Gamma_N) \\
&\quad \geqslant 1 - \exp\{-c'qt\} - \mathbb{P}(\Gamma_N \not\subset K_t^y, T^y = \infty)
\end{aligned}$$

Since

$$\mathbb{P}(\Gamma_N \not\subset K_t^y,\ T^y = \infty) = \mathbb{P}\left(\bigcup_{x \in \Gamma_N} \{x \notin K_t^y,\ T^y = \infty\}\right)$$

it follows from the invariance by translations of the contact process that

$$\begin{aligned}\mathbb{P}(\Gamma_N \not\subset K_t^y,\ T^y = \infty) &= \mathbb{P}\left(\bigcup_{x \in \Gamma_N + y} \{x \notin K_t^0,\ T^0 = \infty\}\right)\\ &\leqslant |\Gamma_N|\ \mathbb{P}(x \notin K_t^0,\ T^0 = \infty)\end{aligned}$$

Durrett and Griffeath[9] obtained some results which, after the breakthrough of Bezuidenhout and Grimmett,[4] are valid for the unrestricted d-dimensional contact process. We use here a result which ensures that there are positive constants a, $\bar{c}$, and $\bar{C}$, independent of N, such that

$$\mathbb{P}(x \notin K_t^0,\ T^y = \infty) \leqslant \bar{C} \exp\{-\bar{c}t\} \qquad \text{for} \quad \|x\| < at \tag{3.1}$$

Note that $\|x\| \leqslant Nd$ for all $x \in \Gamma_N$, and therefore we have

$$\mathbb{P}(\Gamma_N \not\subset K_t^y,\ T^y = \infty) \leqslant |\Gamma_N|\ \bar{C} \exp\{-\bar{c}t\} \qquad \text{for all} \quad t > \frac{Nd}{a}$$

Consequently, there are constants a, c, C such that

$$\begin{aligned}&\mathbb{P}(\xi^\eta(t) = \xi^{\mathbb{Z}^d}(t) \text{ in } \Gamma_N)\\ &\qquad \geqslant 1 - \exp\{-c'qt\} - |\Gamma_N|\ \bar{C} \exp\{-\bar{c}t\}\\ &\qquad \geqslant 1 - |\Gamma_N|\ C \exp\{-ct\}, \qquad \text{for all} \quad t > \frac{Nd}{a}\end{aligned}$$

where $c = \max\{c'q, \bar{c}\} \in (0, \infty)$ and $C = \max\{\bar{C}, 1\} \in [1, \infty)$. Putting $\alpha = \max\{d/a, 1\} \in [1, \infty)$, we get the desired conclusion. ∎

4. PROOF OF THE THEOREM

Following the method employed by Durrett and Schonmann,[10] Mountford[3] has recently shown the following result.

Theorem (Mountford). Let $d \geqslant 1$ and $\lambda > \lambda_c(d)$; we have

$$\frac{T_{\Gamma_N}^{\Gamma_N}}{\mathbb{E}[T_{\Gamma_N}^{\Gamma_N}]} \xrightarrow{\mathscr{L}} \mathrm{EXP}(1) \qquad \text{when} \quad N \to \infty$$

This result is proven using the method introduced in ref. 4 together with ideas on orientated percolation in Durrett.[11]

To state the thermalization of the process, we start by defining, for each $y \in \mathbb{Z}^d$, the translation operators on cylindrical functions by

$$(\tau_y f)(\eta) = f(\eta^{(y)}), \qquad \text{where} \quad \eta^{(y)}(x) = \eta(x-y)$$

Given a cylindrical function f and N, $L \in \mathbb{N}$, with $N > L$, let

$$I_{\Delta(f),N}(L) = \{ y \in \mathbb{Z}^d \colon \Delta(\tau_y f) \subset [-N+L, N-L]^d \cap \mathbb{Z}^d \}$$

and write $\mathbb{E}_\nu(f) = \int f \, d\nu$ to denote the expectation of f with respect to ν.

Theorem 4.1. Let $d \geqslant 1$ and $\lambda > \lambda_c(d)$. Then there is an increasing sequence of positive real numbers $\{b(N, d),\ N \geqslant 1\}$, such that:

(i) We have

$$\frac{b(N,d)}{\mathbb{E}[T_{\Gamma_N}^{\Gamma_N}]} \to 0 \qquad \text{when} \quad N \to \infty$$

(ii) For any $\varepsilon > 0$ and a cylindrical function f, there is $L = L(d, \lambda, \varepsilon, f) \in \mathbb{N}$ such that

$$\mathbb{P}\big(\max_{\substack{l \in \mathbb{Z} \\ 0 \leqslant l < F_N}} \max_{y \in I_{\Delta(f),N}(L)} |A_{b(N,d)}^N(lb(N,d), \tau_y f) - \mathbb{E}_\nu(f)| > \varepsilon \big) \to 0$$

when $N \to \infty$, where

$$F_N = \max\{ l \in \mathbb{N} \colon lb(N,d) < T_{\Gamma_N}^{\Gamma_N} \}.$$

Proof. Since $T_{\Gamma_N}^{\Gamma_N}$ is almost surely finite, for any real, positive number $b(N, d)$, F_N is a well-defined and finite random variable with values in $\mathbb{N}$.

If $b(N, d)$ satisfies condition (i), if follows from Mountford's theorem that

$$\mathbb{P}[F_N = 0] \to 0 \qquad \text{when} \quad N \to \infty$$

Let us now assume that $(b(N, d),\ N \geqslant 1)$ is a sequence satisfying (i). For $\varepsilon > 0$, f cylindrical, $k \in \mathbb{N}$, and $y \in \mathbb{Z}^d$ given, consider the events

$$B_{k,y}^N = [\, |A_{b(N,d)}^N(kb(N,d), \tau_y f) - \mathbb{E}_\nu(f)| > \varepsilon]$$

Then, for any $m \geqslant 1$, $L \geqslant 0$,

$$\begin{aligned}
&\mathbb{P}\left[F_N \geqslant 1, \bigcap_{1 \leqslant k < F_N} \bigcap_{y \in I_{\Delta(f),N}(L)} (B_{k,y}^N)^c\right] \\
&\quad = \sum_{j=1}^{\infty} \left[\mathbb{P}[F_N = j] - \mathbb{P}\left[\bigcup_{k=1}^{j-1} \bigcup_{y \in I_{\Delta(f),N}(L)} B_{k,y}^N, F_N = j\right]\right] \\
&\quad \geqslant \mathbb{P}[1 \leqslant F_N \leqslant m] - \sum_{j=1}^{m} \mathbb{P}\left[\bigcup_{k=1}^{j-1} \bigcup_{y \in I_{\Delta(f),N}(L)} B_{k,y}^N, F_N = j\right] \\
&\quad \geqslant \mathbb{P}[1 \leqslant F_N \leqslant m] \\
&\qquad - \sum_{j=1}^{m} j(2N+1)^d \max_{1 \leqslant k < j} \max_{y \in I_{\Delta(f),N}(L)} \mathbb{P}[B_{k,y}^N, F_N = j] \\
&\quad \geqslant \mathbb{P}[1 \leqslant F_N \leqslant m] \\
&\qquad - m^2(2N+1)^d \max_{j \geqslant 1} \max_{1 \leqslant k < j} \max_{y \in I_{\Delta(f),N}(L)} \mathbb{P}[B_{k,y}^N, F_N = j] \qquad (4.2)
\end{aligned}$$

For any $\eta \in \{0, 1\}^{\mathbb{Z}^d}$, consider K_t^η the set of coupled sites at time t, defined as before by

$$K_t^\eta = \bigcup_{x \in \eta} \{\xi^x(t) \cup (\xi^{\mathbb{Z}^d}(t))^c\} = \xi^\eta(t) \cup (\xi^{\mathbb{Z}^d}(t))^c$$

We have, for $y \in I_{\Delta(f),N}(L)$, that the event

$$[K_t^z \supset [-N+L, N-L]^d \cap \mathbb{Z}^d]$$

for some $z \in [-N+L, N-L]^d \cap \mathbb{Z}^d$, implies (is a subset of)

$$[\tau_y f(\xi_{\Gamma_N}^{\Gamma_N}(t)) = \tau_y f(\xi^{\mathbb{Z}^d}(t))]$$

Choose $L = L(d, \lambda, \varepsilon, f) \in \mathbb{N}$ such that $\exists z \in [-N+L, N-L]^d \cap \mathbb{Z}^d$ with $T^z = \infty$. Fixe one z with the smallest norm. Then we have for $t < T_{\Gamma_N}^{\Gamma_N}$, $k < j$, $y \in I_{\Delta(f),N}(L)$, and $N > L$ the following inequality:

$$\begin{aligned}
&\mathbb{P}[B_{k,y}^N, F_N = j] \\
&\quad = \mathbb{P}[|A_{b(N,d)}^N(kb(N,d), \tau_y f) - \mathbb{E}_\nu(f)| > \varepsilon, F_N = j] \\
&\quad \leqslant \mathbb{P}\left[\left|A_{b(N,d)}^N(kb(N,d), \tau_y f) - \frac{1}{b(N,d)} \int_{kb(N,d)}^{(k+1)\,b(N,d)} \tau_y f(\xi^{\mathbb{Z}^d}(t))\,dt\right| > \frac{\varepsilon}{2} \text{ or}\right. \\
&\qquad \left.\left|\frac{1}{b(N,d)} \int_{kb(N,d)}^{(k+1)\,b(N,d)} \tau_y f(\xi^{\mathbb{Z}^d}(t))\,dt - \mathbb{E}_\nu(f)\right| > \frac{\varepsilon}{2}, F_N = j\right]
\end{aligned}$$

Trajectory by trajectory on $[F_N=j]$, if $k<j$, $y\in I_{A(f),N}(L)$, and $N>L$, we have

$$\left|A^N_{b(N,d)}(kb(N,d),\tau_y f)-\frac{1}{b(N,d)}\int_{kb(N,d)}^{(k+1)\,b(N,d)}\tau_y f(\xi^{\mathbb{Z}^d}(t))\,dt\right|$$

$$\leqslant\frac{2\,\|f\|}{b(N,d)}\int_0^{b(N,d)}\mathbf{1}_{\{\Gamma_{N-L}\not\subset K_t^{\mathbb{Z}},\,T^{\mathbb{Z}}=\infty\}}\,dt,\qquad\text{where}\quad\|f\|=\sup_{\eta\subset\mathbb{Z}^d}f(\eta)$$

Defining the events

$$C^N_{k,y}=\left[\left|\frac{1}{b(N,d)}\int_{kb(N,d)}^{(k+1)\,b(N,d)}\tau_y f(\xi^{\mathbb{Z}^d}(t))\,dt-\mathbb{E}_\nu(f)\right|>\frac{\varepsilon}{2}\right]$$

$$D^{N,L}_k=\left[\frac{\|f\|}{b(N,d)}\int_0^{b(N,d)}\mathbf{1}_{\{\Gamma_{N-L}\not\subset K_t^{\mathbb{Z}},\,T^{\mathbb{Z}}=\infty\}}\,dt>\frac{\varepsilon}{4}\right]$$

$$C^N_k=C^N_{k,0}$$

and as $\mathbb{P}[C^N_{k,y}]$ is independent of y, we have

$$\mathbb{P}[B^N_{k,y},F_N=j]\leqslant\mathbb{P}[C^N_k]+\mathbb{P}[D^{N,L}_k]$$

Inequality (4.2) asserts that condition (ii) is true when condition (i) is, and furthermore we can find a sequence $\{m(N,d),\ N\geqslant 1\}$ such that:

(a) $\mathbb{P}[1\leqslant F_N\leqslant m(N,d)]\to 1$

(b) $m^2(N,d)(2N+1)^d\,(\max_{k\geqslant 1}\mathbb{P}[C^N_k]+\max_{k\geqslant 1}\mathbb{P}[D^{N,L}_k])\to 0$

when $N\to\infty$.

Condition (a) may be written as

$$\mathbb{P}[T^{\Gamma_N}_{\Gamma_N}\leqslant m(N,d)\,b(N,d)]\to 1\qquad\text{when}\quad N\to\infty$$

and using Mountford's theorem, we can write it as

$$\frac{m(N,d)\,b(N,d)}{\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}]}\to\infty\qquad\text{when}\quad N\to\infty$$

Using the notation

$$\psi_L(b(N,d))=\max_{k\geqslant 1}\mathbb{P}[C^N_k]+\max_{k\geqslant 1}\mathbb{P}[D^{N,L}_k]$$

and including part (i) of the theorem, all we have to do now is to find $L \in \mathbb{N}$ and two sequences such that, as $N \to \infty$:

(a) $[m(N, d)\, b(N, d)]/\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}] \to \infty$.

(b) $m^2(N, d)(2N+1)^d\, \psi_L(b(N, d) \to 0$.

(c) $b(N, d)/\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}] \to 0$.

It is easy to show that

$$\lim_{N \to \infty} (\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}])^{-1}(2N+1)^{2d} = 0$$

Therefore if $\psi_L(b(N, d)) \leqslant \bar{C}|\Gamma_N|/b(N, d)$, where $\bar{C} = \bar{C}(\varepsilon, f)$ is a positive constant, we have that

$$m(N, d) = \left[\frac{\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}]}{(2N+1)^{2d}}\right]^{1/5}$$

$$b(N, d) = (\mathbb{E}[T^{\Gamma_N}_{\Gamma_N}])^{9/10}(2N+1)^{d/5}$$

are solutions of (a)–(c) above, concluding the proof. We thus only have to show the next proposition.

Proposition 4.3. Let $d \geqslant 1$, $\lambda > \lambda_c(d)$. For any $\varepsilon > 0$ and f a cylindrical functions, then there are $L = L(d, \lambda, \varepsilon, f) > 0$, $\bar{N} = \bar{N}(\varepsilon, f) > L$, and $\bar{C} = \bar{C}(\varepsilon, f)$ such that

$$\psi_L(b(N, d)) \leqslant \frac{\bar{C}\, |\Gamma_N|}{b(N, d)} \qquad \text{for all} \quad N \geqslant \bar{N}$$

Proof. First we prove that there is a positive constant $C_1 = C_1(\varepsilon, f)$ such that

$$\max_{k \geqslant 1} \mathbb{P}(C^N_k) \leqslant \frac{C_1\, |\Gamma_N|}{b(N, d)}$$

For $k \in \mathbb{N}$, consider the random variables

$$X^b_k = \left|\frac{1}{b}\int_{kb}^{(k+1)b} f(\xi^{\mathbb{Z}^d}(t))\, dt - \mathbb{E}_\nu(f)\right|$$

We have for $k \geqslant 1$ and $b(N, d) > \alpha N$, where $\alpha = \alpha(d, \lambda) \in [1, \infty)$ is given by Lemma 3.2, that

$$\mathbb{E}[X_k^{b(N,d)}]$$

$$\leqslant \frac{2\,\|f\|}{b(N,d)} \int_{kb(N,d)}^{(k+1)b(N,d)} \mathbb{P}(\xi^{\mathbb{Z}^d}(t) \neq \xi^{\nu}(t) \text{ in } \Delta(f))\,dt$$

$$= \frac{2\,\|f\|}{b(N,d)} \int_{kb(N,d)}^{(k+1)b(N,d)} \int_{\{0,1\}^{\mathbb{Z}^d}} \mathbb{P}(\xi^{\mathbb{Z}^d}(t) \neq \xi^{\eta}(t) \text{ in } \Delta(f))\,\nu(d\eta)\,dt$$

Applying Lemma 3.2, it follows that there are $c = c(d, \lambda) \in (0, \infty)$ and $C = C(d, \lambda) \in [1, \infty)$ such that

$$\mathbb{E}[X_k^{b(N,d)}] \leqslant \frac{2\,\|f\|}{b(N,d)} \int_{kb(N,d)}^{(k+1)b(N,d)} C\,|\Gamma_N| \exp\{-ct\}\,dt$$

$$\leqslant \frac{2\,\|f\|\,C\,|\Gamma_N|}{b(N,d)c} \exp\{-kcb(N,d)\} \leqslant \frac{2C'\,\|f\| \cdot |\Gamma_N|}{b(N,d)}$$

for any $k \geqslant 1$, where $C' = C/c$.

Using the Markov inequality, we have that

$$\max_{k \geqslant 1} \mathbb{P}(C_k^N) = \max_{k \geqslant 1} \mathbb{P}\left(X_k^{b(N,d)} > \frac{\varepsilon}{2}\right) \leqslant \max_{k \geqslant 1} \frac{2\mathbb{E}[X_k^{b(N,d)}]}{\varepsilon}$$

$$\leqslant \frac{4C'\,\|f\| \cdot |\Gamma_N|}{\varepsilon b(N,d)} = \frac{C_1(\varepsilon, f)\,|\Gamma_N|}{b(N,d)}$$

where $C_1 = C_1(\varepsilon, f) = 4C'\,\|f\|/\varepsilon$.

The $D_k^{N,L}$ terms can be controlled is an analogous way. For this, choose $L = L(d, \lambda, \varepsilon, f)$ large enough such that there is $z \in [-N+L, N-L]^d \cap \mathbb{Z}^d$, with $T^z = \infty$. Fixe one z with the smallest norm.

For $k \in \mathbb{N}$, let

$$Y_k^{b,L} = \frac{1}{b} \int_0^b \mathbf{1}_{\{\Gamma_{N-L} \not\subset K_t^z,\, T^z = \infty\}}\,dt$$

It follows that

$$\mathbb{E}[Y_k^{b(N,d),L}] = \frac{1}{b(N,d)} \int_0^{b(N,d)} \mathbb{P}(\Gamma_{N-L} \not\subset K_t^z,\, T^z = \infty)\,dt$$

$$= \frac{1}{b(N,d)} \int_0^{b(N,d)} \mathbb{P}\left(\bigcup_{x \in \Gamma_{N-L}} \{x \notin K_t^z,\, T^z = \infty\}\right) dt$$

The translation invariance property of the contact process implies that

$$\mathbb{E}[Y_k^{b(N,d),L}] = \frac{1}{b(N,d)} \int_0^{b(N,d)} \mathbb{P}\left(\bigcup_{x \in z + \Gamma_{N-L}} \{x \notin K_t^0, T^0 = \infty\} \right) dt$$

So we have, for some constant $a \in (0, \infty)$ given, that

$$\begin{aligned}\mathbb{E}[Y_k^{b(N,d),L}] &= \frac{1}{b(N,d)} \int_0^{|\Gamma_{N-L}|/a} \mathbb{P}\left(\bigcup_{x \in z + \Gamma_{N-L}} \{x \notin K_t^0, T^0 = \infty\} \right) dt \\ &\quad + \frac{1}{b(N,d)} \int_{|\Gamma_{N-L}|/a}^{b(N,d)} \mathbb{P}\left(\bigcup_{x \in z + \Gamma_{N-L}} \{x \notin K_t^0, T^0 = \infty\} \right) dt\end{aligned}$$

Using (3.1), there are $a \in (0, \infty)$, $\bar{c} \in (0, \infty)$, and $C \in (0, \infty)$, such that

$$\begin{aligned}\mathbb{E}[Y_k^{b(N,d),L}] &\leqslant \frac{|\Gamma_{N-L}|}{ab(N,d)} + \frac{|\Gamma_{N-L}|}{b(N,d)} \int_{|\Gamma_{N-L}|/a}^{b(N,d)} C \exp\{-\bar{c}t\}\, dt \\ &\leqslant \frac{|\Gamma_{N-L}|}{ab(N,d)} + \frac{C\,|\Gamma_N|}{\bar{c}b(N,d)} \exp\left\{-\frac{|\Gamma_{N-L}|}{a}\right\} \\ &\leqslant \frac{|\Gamma_N|}{ab(N,d)} + \frac{C\,|\Gamma_N|}{\bar{c}b(N,d)} \leqslant \frac{C'\,|\Gamma_N|}{b(N,d)}\end{aligned}$$

where $C' = c' = 2\max\{1/a, C/\bar{c}\}$.

It follows that

$$\begin{aligned}\max_{k \geqslant 1} \mathbb{P}(D_k^{N,L}) &= \max_{k \geqslant 1} \mathbb{P}\left(\|f\|\; Y_k^{b(N,d),L} > \frac{\varepsilon}{4}\right) \\ &\leqslant \max_{k \geqslant 1} \frac{\mathbb{E}[Y_k^{b(N,d),L}]\, 4\, \|f\|}{\varepsilon} \\ &\leqslant \frac{C_2(\varepsilon, f)\, |\Gamma_N|}{b(N,d)}\end{aligned}$$

where $C_2(\varepsilon, f) = C' 4 \|f\|/\varepsilon$. Putting $\bar{C} = \max\{C_1, C_2\}$, the result follows. ∎

5. SUBCRITICAL CASE

We prove now that the theorem of Mountford is false for $\lambda < \lambda_c(d)$.

Theorem 5.1. Let $d \geqslant 1$ and $\lambda < \lambda_c(d)$. For any sequence $\{\gamma_N : N \geqslant 1\}$, $T_{\Gamma_N}^{\Gamma_N}/\gamma_N$ does not converge to an exponential random variable.

Proof. Our method is essentially that employed for Theorem 6 of Schonmann.[2]

We show that

$$\mathbb{P}(T_{\Gamma_N}^{\Gamma_N} \leqslant t \ln N^d) \to 0 \qquad \text{as} \quad N \to \infty \qquad \text{if} \quad t < 1$$

and that there exists $K = K(\lambda) > 1$ such that

$$\mathbb{P}(T_{\Gamma_N}^{\Gamma_N} \leqslant t \ln N^d) \to 1 \qquad \text{as} \quad N \to \infty \qquad \text{if} \quad t > K$$

The first part follows from the fact that

$$T_{\Gamma_N}^{\Gamma_N} \geqslant S_{\Gamma_N} = \max_{x \in \Gamma_N} \mathcal{U}_1^x$$

where $\mathcal{U}_1^x$ is the instant of the first occurrence of the Poisson process with rate one defined in $x \in \mathbb{Z}^d$. So

$$\begin{aligned}\mathbb{P}(T_{\Gamma_N}^{\Gamma_N} \leqslant t \ln N^d) &\leqslant \mathbb{P}(S_{\Gamma_N} \leqslant t \ln N^d)\\ &= [1 - \exp\{-t \ln N^d\}]^{(2N+1)^d}\\ &= \left[1 - \frac{1}{(2N+1)^d} \frac{(2N+1)^d}{N^{td}}\right]^{(2N+1)^d} \to 0\end{aligned}$$

as $N \to \infty$ if $t < 1$.

To prove the second part, we use the result (1.13) in Bezuidenhout and Grimmett.[12] It defines the distance function

$$\delta((x, t), (y, s)) = |t - s| + \sum_{i=1}^{d} |x_i - y_i| \qquad \text{for} \quad (x, t), (y, s) \in \mathbb{Z}^d \times \mathbb{R}$$

For $r > 0$ the ball $S(r) = \{\pi \in \mathbb{Z}^d \times \mathbb{R}\colon \delta(0, \pi) \leqslant r\}$ and its surface $\partial S(r) = \{\pi \in \mathbb{Z}^d \times \mathbb{R}\colon \delta(0, \pi) = r\}$ are also defined. Considering those definitions, it is then proved that for $\lambda < \lambda_c(d)$ there exists $\psi = \psi(\lambda) > 0$ such that

$$\mathbb{P}\{0 \to \partial S(r)\} \leqslant \exp\{-r\psi(\lambda)\} \qquad \text{for all} \quad r$$

where $A \to B$ means that there exist $a \in A$ and $b \in B$ such that a and b are in the same connected component of a (random) graph lying entirely within $\mathbb{R}^{d+1}$. See Section 2.1 of ref. 12 for details, which include the topology used and the definition for an event to be determined by the configuration inside a region of $\mathbb{R}^{d+1}$.

For $t > K = \max\{1, [\psi(\lambda)]^{-1}\}$ we have that

$$\begin{aligned}
\mathbb{P}\{T_{\Gamma_N}^{\Gamma_N} > t \ln N^d\} &\leqslant \mathbb{P}\{\xi^{\Gamma_N}(K \ln N^d) \neq \varnothing\} \\
&\leqslant |\Gamma_N| \, \mathbb{P}\{\xi^0(K \ln N^d) \neq \varnothing\} \\
&\leqslant |\Gamma_N| \, \mathbb{P}\{0 \to \partial S(K \ln N^d)\} \\
&\leqslant |\Gamma_N| \exp\{-(K \ln N^d) \, \psi(\lambda)\} \\
&= \frac{|\Gamma_N|}{N^{Kd\psi(\lambda)}} \to 0 \qquad \text{as} \quad N \to \infty \quad \blacksquare
\end{aligned}$$

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